Chaos in small-world networks

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A nonlinear small-world network model has been presented to investigate the effect of nonlinear interaction and time delay on the dynamic properties of small-world networks. Both numerical simulations and analytical analysis for networks with time delay and nonlinear interaction show chaotic features in the system response when nonlinear interaction is strong enough or the length scale is large enough. In addition, the small-world system may behave very differently on different scales. Time-delay parameter also has a very strong effect on properties such as the critical length and response time of small-world networks.

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I. INTRODUCTION

Since the pioneer work of Watts and Strogatz [1] on small-world networks, a lot of interesting research on the theory and application of small-world networks [2–7] have been initiated. The properties of complicated networks such as internet servers, power grids, forest fires, and disordered porous media are mainly determined by the way of connections between the vertices or occupied sites. One limiting case is the regular network with a high degree of local clustering and a large average distance, while the other limiting case is the random network with negligible local clustering and a small average distance. The small-world network is a special class of networks with a high degree of local clustering as well as a small average distance. Such small-world phenomenon can be obtained by adding randomly only a small fraction of the long-range connections, and some common networks such as power grids, financial networks and neural networks behave similar to small-world networks [2–9].

The dynamic features such as spreading and response of an influence over a network have also been investigated in recent studies [2,3] by using shortest paths in system with sparse long-range connections in the frame work of small-world models. A simple time-stepping rule has been used to simulate the spreading of the influence such as a forest fire, an infectious disease, or a particle in percolating media. The influence propagates from the infected site to all uninfected sites connected to it via a link at each time step, whenever a long-range connection or shortcut is met, the influence is newly activated at the other end of the shortcut so as to simulate long-range sparkling effect such as the infect site (e.g., a person with influenza) suddenly travels to a new place, or an infected portable computer starts to connect to a network at a new site. These phenomena have been successfully studied by the Newman and Watts model [2] and Moukarzel [3]. Their models are linear model in the sense that the governing equation is linear and the response is immediate as there is no time delay in their models.

However, in reality, a spark or an infection cannot start a new fire spot or new infection immediately, it usually takes some time Δ, called ignition time or waiting time, to start a new fire or infection. In addition, a fraction of infected sites shall recover after a further time of T to normality. Thus the existing models are no longer be able to predict the response in the networks or systems with a time delay. Furthermore, the nonlinear effect such as the competition factor as in the population dynamics, congestion features such as the traffic jam in internet communication and road networks, and the frictional or viscous effect in the interaction of vertices, shall be modelled in order to simulate more realistic networks. When considering these nonlinear effects, the resulting small-world network model is generally no longer linear. Therefore, a nonlinear model is yet to be formulated.

The main aim of this paper is to present a more general nonlinear model for the small-world networks by extending the existing Newman-Watts [2] and Moukarzel [3] models to investigate the effects of time delay, site recovery, and the nonlinear interaction due to competition and congestion. The new model will generally lead to a nonlinear difference differential equation, whose solution is usually very difficult to obtain if it is not impossible. Thus the numerical simulation becomes essential [1]. However, we will take the analytical analysis as far as possible and compare with the results from numerical simulations. The characteristic chaos of the network dynamics is then studied by reducing the governing equation into a logistic equation. The control of the chaos is also investigated by introducing the negative feedback with a time delay to the small-world networks.

II. NONLINEAR MODEL FOR SMALL-WORLD NETWORKS

To investigate the nonlinear effect of the time delay on the properties of a small-world network, we now consider a randomly connected network on a d-dimensional lattice [1,2] (with $d=1,2,\ldots$), and overlapping on the network are a number of long-range shortcuts randomly connecting some vertices, and the fraction of the long-range shortcuts or probability p is relative small ($p \ll 1$). Now assuming an influence or a pollutant particle spreads with a constant velocity $u=1$ in all directions and a newly infected site in the other end of a shortcut will start but with a time delay Δ. Following the method developed by Newman and Watts [2] and Moukarzel [3], the total influenced volume $V(t)$ comes from

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three contributions: one is the influenced volume with $\Gamma_d\int_0^t \zeta^{d-1} d\zeta$ where t is time and Γ_d is a shape factor, the other contribution is $\Gamma_d\int_0^t [2pV(t-\zeta-\Delta)]\zeta^{d-1}d\zeta$ for a hypersphere started at time ζ. These two components have been used earlier [2,3] although without the time delay parameter. Now we add the third component due to the nonlinear interactions such as friction, slow down due to the congestion as in the case of internet network and traffic jam and lack of other resource such as insufficient oxygen for the fire spark to start a new fire. By assuming this nonlinear effect as $-\Gamma_d\int_0^t[\mu V^2(t-\zeta-\Delta)]\zeta^{d-1}d\zeta$ where $\mu \ll 1$ is a measure of nonlinear interaction, a continuum approach to the network leads to the following delay equation

$$V(t)=\Gamma_d\int_0^t \zeta^{d-1}[1+\xi^{-d}V(t-\zeta-\Delta)-\mu V^2(t-\zeta-\Delta)]d\zeta, \tag{2.1}$$

where $d=1,2,\ldots,$ and Γ_d is shape factor of a hypersphere in d dimensions. The *Newman-Watts* length scale [2] can be conveniently defined as

$$\xi=\frac{1}{(2pkd)^{1/d}}, \tag{2.2}$$

where $k=$const is some fixed range. Rescaling t by

$$\tau=t[\Gamma_d\xi^{-d}(d-1)!]^{1/d}, \quad \delta=\Delta[\Gamma_d\xi^{-d}(d-1)!]^{1/d} \tag{2.3}$$

and rewriting Eq. (2.1) in the rescaled form

$$V(t)=\frac{\xi^d}{(d-1)!}\int_0^\tau(\tau-\zeta)^{d-1}[1+\xi^{-d}V(\zeta-\delta) - \mu V^2(\zeta-\delta)]d\zeta. \tag{2.4}$$

After differentiating the equation d times, we have

$$\frac{d^dV}{d\tau^d}=\xi^d+V(\tau-\delta)-\mu\xi^dV^2(\tau-\delta), \tag{2.5}$$

which is a nonlinear delay differential equation, whose explicit solutions is not always possible. It is worth pointing out that the present model can degenerate into the previous simplified mode by Newman and Watts [2] and Moukarzel [3] when $\mu=0$ and $\delta=0$, which produces exponential growth without limit. However, in reality, the nonlinear interaction due to competition exists as in the case of population dynamics where each individual compete for food. Thus, μ is generally nonzero, and this infected volume V is the results of competitive balance between growth term and nonlinear interaction terms. Similar to the population dynamics, the system with all initial conditions eventually settles into one of three different types of behavior: fixed state, periodic, and chaotic, depending on the parameters of μ and ξ. In addition, the time delay (δ) can also have strong effect on the dynamic properties of the small-world networks.

III. CHAOS IN SMALL-WORLD NETWORKS

From the theory of dynamical systems, it is expected that the dynamic features can be shown more clearly by using the representation in Poincare plane [10], which usually transforms a nonlinear differential equation into a nonlinear iterated map or logistic equation. Now we write Eq. (2.5) in a difference form and take $d\tau=\delta$ to get a logistic equation. In order to focus on the main characteristics of the dynamics, for simplicity, we can take $\delta=1$ in one dimension ($d=1$), and we then have

$$V_{n+1}=\xi+2V_n-\mu\xi V_n^2, \tag{3.1}$$

where $V_{n+1}=V(\tau)$ and $V_n=V(\tau-1)$. By changing variables

$$v_{n+1}=\frac{\mu\xi}{(2+2A\mu\xi)}(V_{n+1}+A),$$
$$v_n=\frac{\mu\xi}{(2+2A\mu\xi)}(V_n+A), \quad A=\frac{\sqrt{1+4\mu\xi^2}-1}{2\mu\xi}, \tag{3.2}$$

we can rewrite Eq. (3.1) as

$$v_{n+1}=\lambda v_n(1-v_n), \quad \lambda=(\sqrt{1+4\mu\xi^2}+1), \tag{3.3}$$

which is a standard form of the well-known logistic equation [10]. This is a well-studied logistic equation and the parameter range of λ for period doubling and chaos is well known. Thus, we can express the length scale ξ in terms of λ as

$$\xi^2=\frac{(\lambda-1)^2-1}{4\mu}. \tag{3.4}$$

The system becomes chaotic as λ is bigger than $\lambda_* \approx 3.5699$ but usually below 4.0, so the chaos begins at

$$\xi_*=\sqrt{\frac{1.401}{\mu}}. \tag{3.5}$$

For λ less than $\lambda_0 \approx 3.0$, the system approach to a fixed point, that is,

$$\xi_0=\sqrt{\frac{0.75}{\mu}}. \tag{3.6}$$

For a fixed μ, when $\xi_0<\xi<\xi_*$, then $\lambda<\lambda_*$, the system is in a period doubling cascade. When $\xi>\xi_*$, the system becomes chaotic. Clearly, as $\mu\to 0$, $\xi_*\to\infty$. The system behavior depends on the length scale of small-world networks. The system may look chaotic on a large scale greater than the critical length scale ξ_* and the same system may be well regular on the even smaller scale. So the system behaves differently on different scales.

On the other hand, for a fixed length scale ξ, we can define a critical value of μ_* when $\lambda=\lambda_*$

$$\mu_*=\frac{1.401}{\xi^2}. \tag{3.7}$$

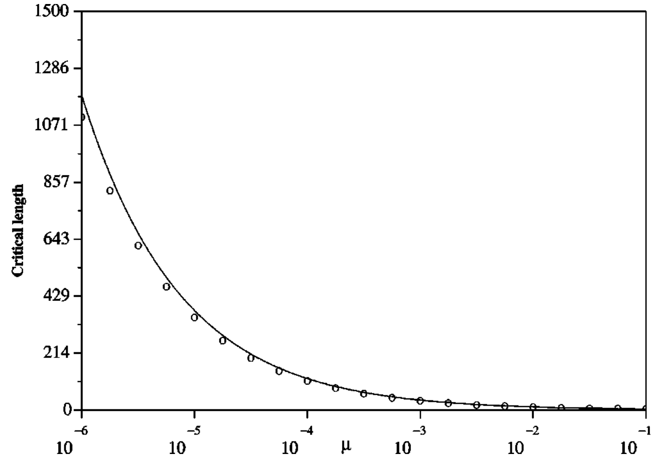

FIG. 1. Critical length versus the nonlinear interaction coefficient μ for a network size $N=500\,000$ and $p=0.002$. All the variables are dimensionless. Numerical results (marked with open circles) agree well with analytical express (solid).

For weak competition or nonlinear interaction $\mu<\mu_*$, then $\lambda<\lambda_*$, so that the system falls into the period doubling cascade. For the case of strong competition $\mu>\mu_*$, $\lambda>\lambda_*$, the system becomes chaotic. This clearly shows that for a given size of networks, too much strong competition or nonlinear interaction can make the system chaotic. This can have important implications in social sciences and financial networks. Weak competition can provide the markets variety while too much competition could cause chaos if it is not properly controlled.

To check the analytical results, we have also simulated the scenario by using the numeric method [1,2] for a network size $N=500{,}000$, $p=0.002$, and $k=2$ on a one-dimensional lattice. Different values of the nonlinear interaction coefficient μ are used and the related critical length ξ_* when the system of small-world networks becomes chaotic. Figure 1 shows ξ_* for different values of μ where all values are nondimensional. The solid curve is the analytical results (3.5) and the points (marked with open circle) are numerical simulations. The good agreement verifies the analysis. However, as the typical length increases, the difference between these two curves becomes larger because the governing equation is mainly for infinite size network. So the difference is due to the finite size of the network used in the simulations.

IV. NEGATIVE FEEDBACK AND CHAOS CONTROL OF SMALL-WORLD NETWORKS

The occurrence of the chaos in small-world networks is due to the nonlinear interaction term with a time delay. This chaotic feature can be controlled by adding a negative feedback term [11,12]. In reality, the influence such as a signal or an influence (e.g., influenza) only last a certain period of time T, then some of the influenced sites recover to normality. From the derivation of small-world model equation (2.1), we see that this adds an extra term $\beta V(t-\Delta-T)$, which means that a fraction (β) of the infected sites at a much earlier time $(t-\Delta-T)$ shall recover at t. So that we have the modified form of Eq. (2.5) as

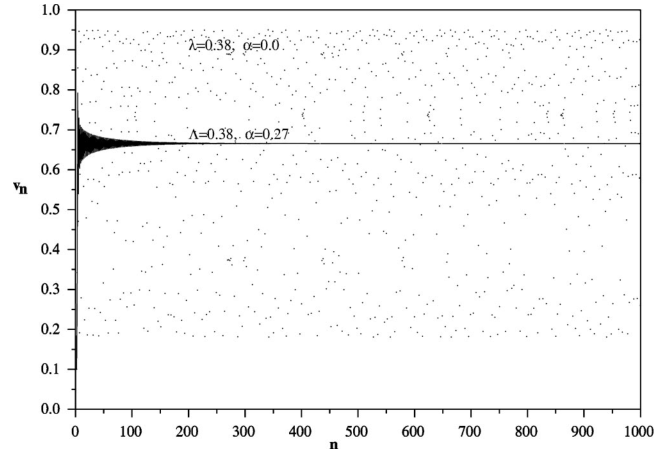

FIG. 2. Comparison of chaos and chaos control due to a delay feedback. All variables and parameters are dimensionless. The dotted points are for the chaotic response when there is no feedback $(\Lambda=3.8,\ \alpha=0)$, while the solid curve corresponds to the just control of the chaos by a negative feedback $(\Lambda=3.8,\ \alpha=0.27)$.

$$\frac{d^dV}{d\tau^d}=\xi^d+V(\tau-\delta)-\mu\xi^dV^2(\tau-\delta)-\beta\xi^dV(\tau-\delta-\tau_0), \tag{4.1}$$

where $\tau_0=T[\Gamma_d\xi^{-d}(d-1)!]^{1/d}$. For $d=1$, we can take $\tau_0=j\delta$ $(j=1,2,\dots)$ without losing its physical importance. By using transform (3.2), we have a modified logistic equation

$$v_{n+1}=\lambda v_n(1-v_n)+\alpha(v_n-v_{n-j}),\quad j=1,2,\dots, \tag{4.2}$$

with

$$\lambda=(\sqrt{1+4\mu\xi^2}+1),\quad \alpha=\beta\xi, \tag{4.3}$$

which is in agreement with the Escalona and Parmananda form [12] of the OGY algorithm [11] in the chaos control strategy. We can also write Eq. (4.2) as

$$v_{n+1}=\Lambda v_n(1-v_n)-\alpha v_{n-j},\quad j=1,2,\dots \tag{4.4}$$

with

$$\Lambda=[\sqrt{(1-\beta\xi)^2+4\mu\xi^2}+1],\quad \alpha=\beta\xi. \tag{4.5}$$

This last form (4.4) emphases the importance of the time delay and the effect of negative feedback in controlling the chaos.

For a fixed value of $\Lambda=3.8$, we find a critical value of $\alpha_*=0.27$ for $j=1$ and $\alpha_*=0.86$ for $j=2$ to just control the chaos so that the system settles to a fixed point. For the case of $\alpha>\alpha_*$, the feedback is so strong that the chaos is substantially controlled, and the system leads to a fixed state very quickly, and the numerical simulation shows that all the estimated Lyapunov exponent is nonpositive or $L_\lambda\approx(1/N)\sum_{n=1}^{N}\log_2|\Lambda-2\Lambda v_n|\leqslant 0$. For $\alpha<\alpha_*$, the feedback is not strong enough and the chaos is not substantially suppressed. The chaos-measuring Lyapunov exponent estimated from the numerical simulations is usually non-negative, which usually means that system is chaotic for weak feedback. Figure 2 shows that the effect of recovery of the infected site or the delay feedback on the system behavior (for $N=1000$). The dotted points are for the chaotic response when there is no feedback ($\Lambda=3.8$, $\alpha=0$), while the solid curve corresponds to the just control of the chaos by a negative feedback ($\Lambda=3.8$, $\alpha=0.27$). This clearly indicates that the proper feedback due to healthy recovery and time delay can control the chaotic response to a stable state.

V. CONCLUSION

A nonlinear small-world network model has been presented here to characterize the effect of nonlinear interactions, time delay, and recovery on small-world networks. Numerical simulations and analytical analysis for networks with a time delay and nonlinear interactions show that the system response of the small-world networks may become chaotic on the scale greater than the critical length scale ξ_*, and at the same time the system may still behave regularly on a smaller scale. So the small-world system behaves differently on different scales. The time delay parameter δ has a very strong effect on properties such as the critical length and response time of the networks.

On the other hand, in order to control the possible chaotic behavior of small-world networks, a proper feedback or healthy recovery of the infected sites is needed to stable the system response. For a negative delay feedback, numerical simulations suggest that a linear recovery rate β or linear feedback can properly control the chaos if the feedback is strong enough. This may have important applications in the management and control of the dynamic behavior of the small-world networks such as the financial and business networks and world wide webs. This shall be the motivation of some further studies of the dynamics of small-world networks.

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[1] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[2] M. E. J. Newman and D. J. Watts, Phys. Rev. E **60**, 7332 (1999).
[3] C. F. Moukarzel, Phys. Rev. E **60**, R6263 (1999).
[4] M. E. J. Newman, C. Moore, and D. J. Watts, Phys. Rev. Lett. **84**, 3201 (2000).
[5] A. Barrat and M. Weigt, Eur. Phys. J. B **13**, 547 (2000).
[6] B. Bollobas, *Random Graphs* (Academic Press, New York, 1985).
[7] S. A. Pandit and R. E. Amritkar, Phys. Rev. E **60**, R1119 (1999).
[8] M. Barthelemy and Luis A. Nunes Amaral, Phys. Rev. Lett. **82**, 3180 (1999).
[9] D. J. Watts, *Small Worlds: The Dynamics of Networks Between Order and Randomness* (Princeton University Press, Princeton, 1999).
[10] F. C. Moon, *Chaotic and Fractal Dynamics* (Wiley, New York, 1992).
[11] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[12] J. Escalona and P. Parmananda, Phys. Rev. E **61**, 2987 (2000).